

# LOCAL CONTROL OF (4,5,7,8-10)–FILTRATION SNAKE ROBOT VIA CGA

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**Abstract:** *We describe the local control of a (6–8)–link snake like robot endowed with omnidirectional wheels on two links (head and tail). All calculations including the position, direct kinematics, differential kinematics and inverse kinematics are described in the terms of CGA only.*

**Keywords:** *Robotic snake, conformal geometric algebra, nonholonomic kinematics*

## 1 Introduction

Let a non–holonomic system be described by the appropriate dynamic (Pfaff) system of ODEs, which defines the vectors of admissible motion directions (w.r.t. the controlling parameters). These vectors together with the Lie bracket operation generate a Lie algebra corresponding to the original system. Consequently, a non–holonomic system is said to be locally controllable if the linear span of its appropriate Lie algebra generators is of the same dimension as the configuration space (Rachevsky–Chow Theorem). Furthermore, while generating the Lie algebra, one obtains a natural filtration (w.r.t. the number of Lie bracket applications). The elements of the higher filtration parts correspond to the motions, which can not be realized locally but are necessary for local controllability. A general interpretation of such motions is crucial for local controllability description and realization. The main goal of the talk is to employ an advanced mathematical tool of the Conformal Geometric Algebra (CGA) in the snake robot modeling in order to achieve the optimal control of particular solutions.

## 2 Conformal geometric algebra – CGA

Let  $\mathbb{R}^{4,1}$  denote a vector space  $\mathbb{R}^5$  equipped with the scalar product of signature (4, 1), let  $Cl(4, 1)$  denote the corresponding Clifford algebra, i.e. a free, associative and distributive algebra as a span of the set  $\{e_1, e_2, e_3, e_+, e_-\}$  such that the following identities are satisfied:

$$e_1^2 = e_2^2 = e_3^2 = e_+^2 = 1, \quad e_-^2 = -1,$$

$$e_i e_j = -e_j e_i, \quad i \neq j, \quad i, j \in \{1, 2, 3, +, -\}.$$

In this case, we get  $2^5 = 32$ –dimensional vector space. Let us note that the norm in  $\mathbb{R}^{4,1}$  can be understood as a vector square  $x^2 = \|x\|^2$ . Now, we define two additional products on  $\mathbb{R}^{4,1}$  based on the geometric one for any  $u, v \in \mathbb{R}^{4,1}$ , *dot product* and *wedge product*, respectively:

$$u \cdot v = \frac{1}{2}(uv + vu), \quad u \wedge v = \frac{1}{2}(uv - vu)$$

and thus the formula for the geometric product can be derived as

$$uv = u \cdot v + u \wedge v.$$

Generally, the wedge (outer) product of two basis blades  $E_i$  and  $E_j$ , with  $k = \text{gr}(E_i)$ ,  $l = \text{gr}(E_j)$  is defined as

$$E_i \wedge E_j := \langle E_i E_j \rangle_{k+l}$$

and the dot (inner) product is defined as

$$E_i \cdot E_j = \begin{cases} \langle E_i E_j \rangle_{|k-l|} & i, j > 0 \\ 0 & i = 0 \text{ or } j = 0, \end{cases}$$

where  $\text{gr}(E)$  is a grade of the basis blade  $E$  and  $\langle \cdot \rangle_k$  is the grade projection into the blade of the grade  $k$ . To work with CGA effectively, we have to define a new basis of  $\mathbb{R}^{4,1}$  as a set  $\{e_1, e_2, e_3, e_0, e_\infty\}$  such that  $e_0 = \frac{1}{2}(e_- + e_+)$  and  $e_\infty = (e_- - e_+)$ . Consequently, the following properties hold:

$$e_0^2 = 0, \quad e_\infty^2 = 0, \quad e_\infty e_0 = -1 - e_- \wedge e_+, \quad e_0 e_\infty = -1 + e_- \wedge e_+, \\ e_\infty e_0 = -e_0 e_\infty - 2.$$

In CGA, we can represent the basis geometric elements by the following multivectors from  $\mathcal{Cl}(4, 1)$ :

$$\begin{aligned} \text{point } x &\rightsquigarrow Q = x + \frac{1}{2}x^2 e_\infty + e_0 \\ \text{sphere of radius } r &\text{ and center } C \rightsquigarrow S = C - \frac{1}{2}r^2 e_\infty \\ \text{point pair } Q_1, Q_2 &\rightsquigarrow P = Q_1 \wedge Q_2. \end{aligned}$$

In CGA (in fact in GA generally), any transformation of the element  $O$  is realized by conjugation

$$O \mapsto MO\tilde{M}$$

where  $M$  is the appropriate multivector from  $\mathcal{Cl}(4, 1)$ . For instance, the translation in the direction  $t = t_1 e_1 + t_2 e_2 + t_3 e_3$  is realized by the multivector

$$M := T = 1 - \frac{1}{2}t e_\infty$$

and the rotation around the axis  $L$  by angle  $\phi$  is realized by the multivector

$$M := R = \cos \frac{\phi}{2} - L \sin \frac{\phi}{2}$$

where  $L = a_1 e_2 e_3 + a_2 e_1 e_3 + a_3 e_1 e_2$ . For more information refer to books [2, 8] or papers [3, 4].

### 3 The flag structure

The snake robot described in this paper consists of 6–8 rigid links of constant length 2 interconnected by motorized joints. Except for the first and the last, to each line, in the center of mass, a pair of wheels is attached to provide an important snake-like property that the ground friction in the direction perpendicular to the link is considerably higher than the friction of a simple forward move. In particular, this prevents the slipping sideways. To determine the actual position of a snake robot we need the set of 8–10 generalized coordinates

$$q = (x, y, \theta, \Phi_i, i \in \{1, \dots, N\}) \in M, \tag{1}$$

where  $N \in \{5, 6, 7\}$  and  $M = \mathbb{R}^2 \times (\mathbb{S}^1)^{N+1}$  which describes the configuration of the snake robot.

The control theory generally leads to a nonholonomic system

$$\dot{q} = \sum_{i=1}^m u_i X_i(q), \quad q \in M,$$

where  $X_i$  are vector fields on  $M$ , where  $m \in \mathbb{N}, m < \dim M$ . In particular, the controllability of the nonholonomic system is fully characterized by the properties of the Lie algebra generated by  $X_1, \dots, X_m$ . We define

$$\Delta^1 = \text{span}\{X_1, \dots, X_m\}$$

and  $\Delta^{s+1} = \Delta^s + [\Delta^1, \Delta^s]$ , where  $[\Delta^1, \Delta^s] = \text{span}\{[X, Y] : X \in \Delta^1, Y \in \Delta^s\}$ . The Lie algebra  $\bar{\Delta}$  generated by  $X_1, \dots, X_m$  is defined as

$$\bar{\Delta} = \bigcup_{s \geq 1} \Delta^s.$$

Let us note that our system satisfies Chow's condition, i.e.  $\bar{\Delta}(q) = T_q M, \forall q \in M$  and the vectors at  $q \in M$  form a flag of subspaces of  $T_q M$ , that is

$$\Delta^1(q) \subset \Delta^2(q) \subset \dots \subset \Delta^{r-1}(q) \subset \Delta^r(q) = T_q M,$$

where  $r = r(p)$  is so-called degree of nonholonomy at  $p$ . Set  $n_i(q) = \dim \Delta^i(q)$ . The  $r$ -tuple of integers  $(n_1(q), \dots, n_r(q))$  is called the growth vector at  $q$ . In our cases the growth vectors are

$$(4, 5, 7, 8), (4, 5, 7, 9), (4, 5, 7, 10)$$

and the degree of nonholonomy is 4. The structure of the flag may also be described by another sequence of integers. We define the weights at  $q$ ,  $w_i = w_i(q)$ ,  $i = 1, \dots, m$ , by setting  $w_j = s$  if  $n_{s-1}(q) < j \leq n_s(q)$ , where  $n_0 = 0$ . In our cases the weights are

$$(1, 1, 1, 1, 2, 3, 3, 4), (1, 1, 1, 1, 2, 3, 3, 4, 4), (1, 1, 1, 1, 2, 3, 3, 4, 4, 4).$$

As the first and the last link are endowed with the omnidirectional wheels and do not affect the control, the controlling Lie algebra is of a special form, particularly it contains a two-dimensional center  $\mathcal{Z}$ . The flag structures on factor space  $\bar{\Delta}/\mathcal{Z}$  are characterized by the growth vectors

$$(2, 3, 5, 6), (2, 3, 5, 7), (2, 3, 5, 8),$$

where the last example is of full dimension as the corresponding P. Hall basis contains exactly 8 elements. As a result, we discuss all possibilities with the degree of nonholonomy equal to 4 after the center factorisation.

### 4 Kinematics

Note that a fixed coordinate system  $(x, y)$  is attached. The points  $p_i := (x_i, y_i)$ , denote the centers of mass of each link. To describe the robotic snake we use as a central object the set of point pairs

$$(P_0, P_2, P_4, \dots, P_N)$$

where  $P_0 = Q_0 \wedge Q_1$ ,  $P_1 = Q_1 \wedge Q_2$  and  $P_N = Q_N \wedge Q_{N+1}$ , where  $Q_i$  are the joint points and head and tail points. Consequently, the kinematic equations can be assessed and if we consider the projections

$$Q_i = -\frac{\sqrt{P_i \cdot P_i} + P_i}{e_\infty \cdot P_i}, \quad Q_{i+1} = \frac{\sqrt{P_i \cdot P_i} + P_i}{e_\infty \cdot P_i},$$

we are able to express the coordinates of every point from any point pair. The coordinates of particular position of link centers are the following

$$p_i = P_i e_\infty \tilde{P}_i, \text{ s.t. } P_i = R_{\Phi_i} \cdots R_{\Phi_1} R_\theta T_{x,y} P_{i,0} \tilde{T}_{x,y} \tilde{R}_\theta \tilde{R}_{\Phi_1} \cdots \tilde{R}_{\Phi_i}$$

and, for the robotic snake initial position  $x = y = \theta = \Phi_i = 0$ , the appropriate point pairs are denoted by  $P_{i,0}$  and calculated directly, e.g.

$$\begin{aligned} P_{1,0} &= (e_0) \wedge (2e_1 + 2e_\infty + e_0) = 2e_0e_1 - 2e_+e_-, \\ P_{2,0} &= (2e_1 + 2e_\infty + e_0) \wedge (4e_1 + 8e_\infty + e_0) = 2e_0e_1 + 8e_1e_\infty - 6e_+e_-, \\ P_{3,0} &= (4e_1 + 8e_\infty + e_0) \wedge (6e_1 + 18e_\infty + e_0) = 2e_0e_1 + 24e_1e_\infty - 10e_+e_- \\ &\vdots \end{aligned}$$

Now, the transformations corresponding to the generalized coordinates can be written as

$$\begin{aligned} T_{x,y} &= 1 - \frac{1}{2}(xe_1 + ye_2)e_\infty, \\ T_{Q_i} &= 1 - \frac{1}{2}Q_i e_\infty, \\ R_0 &= \cos \frac{\theta}{2} - L_0 \sin \frac{\theta}{2}, \text{ where } L_0 = T_{x,y} e_1 e_2 \tilde{T}_{x,y}, \\ R_i &= \cos \frac{\Phi_1}{2} - L_i \sin \frac{\Phi_1}{2}, \text{ where } L_i = T_{Q_i} e_1 e_2 \tilde{T}_{Q_i}. \end{aligned}$$

The direct kinematics for the snake robot is obtained similarly as the kinematics for serial robot arms [10]. In general, it is given by a succession of generalised rotations  $R_i$  and it is valid for all geometric objects, including point pairs. A point pair  $P$  in a general position is computed from its initial position  $P_0$  as follows

$$P = \prod_{i=1}^n R_i P_0 \prod_{i=1}^n \tilde{R}_{n-i+1}. \tag{2}$$

## 5 Differential kinematics

Unlike the fixed serial robot arms, we allow  $R_i$  to be also a translation. We view translations as degenerate rotations. Then the differential kinematics is expressed by means of the total differential as follows

$$dP = \sum_{j=1}^n \partial_{q_j} \left( \prod_{i=1}^n R_i P_0 \prod_{i=1}^n \tilde{R}_{n-i+1} \right) dq_j.$$

**Theorem.** *Let  $c$  be a centre of a sphere  $S$  (including a point pair as a  $0D$ -sphere) whose final position is given by the kinematic chain (2). Then the differential kinematics of  $c$  is given by*

$$\dot{c} = \sum_{j=1}^n [c \cdot L'_j] dq_j.$$

*Proof.* See [4]. □

Concretly, we obtain the system

$$\begin{aligned} \dot{p}_1 &= [p_1 \cdot e_1 e_\infty] \dot{x} + [p_1 \cdot e_2 e_\infty] \dot{y} + [p_1 \cdot L_0] \dot{\theta}, \\ \dot{p}_i &= [p_2 \cdot e_1 e_\infty] \dot{x} + [p_2 \cdot e_2 e_\infty] \dot{y} + [p_i \cdot L_0] \dot{\theta} + [p_2 \cdot L_1] \dot{\Phi}_1 + \dots + [p_2 \cdot L_1] \dot{\Phi}_i, \end{aligned}$$

which in the matrix notation is of the form

$$\dot{p} = J \dot{q}, \quad (3)$$

where  $q$  are our coordinates (1) and  $J = (j_{kl})$  is a matrix with the elements defined by scalar products of points and axes

$$\begin{aligned} j_{i1} &= [p_i \cdot e_1 e_\infty], \quad j_{i2} = [p_i \cdot e_2 e_\infty], \\ j_{ik} &= [p_i \cdot L_{k-3}] \text{ for } 3 \leq k < 3 + i, \\ j_{ik} &= 0 \text{ for } 3 + i \leq k. \end{aligned}$$

As the wheels do not slip to the side direction, the velocity constraint condition is satisfied for each link  $i$  and in terms of CGA can be written as

$$\dot{p}_i \wedge P_i \wedge e_\infty = 0. \quad (4)$$

Thus if we substitute (3) in (4), we obtain a system of linear ODEs, which has a simple Pfaff matrix form

$$A \dot{q} = 0, \quad (5)$$

where  $A = (a_{ij})$  is a matrix with the elements defined by

$$a_{ik} = j_{ik} \wedge P_i \wedge e_\infty. \quad (6)$$

Note that the enteries of  $A$  are multiples of  $e_1 e_2 e_+ e_-$ . Taking the conjugate and multiplying with  $e_3$   $A$  can be considered simply as a matrix over the field of functions. For example, the solution of this system with respect to  $\dot{\theta}$  parameterized by  $\dot{x}, \dot{y}$ , (i.e.  $\dot{x} = t_1$  and  $\dot{y} = t_2$ ) is of the form

$$\dot{\theta} = - \frac{[p_1 \cdot e_1 e_\infty] \wedge P_1 \wedge e_\infty}{[p_1 \cdot L_0] \wedge P_1 \wedge e_\infty} t_1 - \frac{[p_1 \cdot e_2 e_\infty] \wedge P_1 \wedge e_\infty}{[p_1 \cdot L_0] \wedge P_1 \wedge e_\infty} t_2.$$

If we denote by  $e^*$  the dual to  $e$  in CGA which is realized by the multiplication of the inverse unit pseudoscalar, the straightforward computation leads to  $[p_1 \cdot L_0] \wedge P_1 \wedge e_\infty = 2e_3^*$ , i.e. the solution always exists, because  $([p_1 \cdot L_0] \wedge P_1 \wedge e_\infty)^{-1} = -\frac{1}{2}e_3^*$ .

The singular posture of the system is in the case that the wheel axes, i.e. lines perpendicular to each link containing the link center point, intersect in precisely one point or are parallel, see Figure 2. In our setting this is one condition only because in CGA the parallel lines intersect in exactly one point which is  $e_\infty$ . It is

easy to see that this happens in such case that all joints lie on a single circle, i.e. in CGA they satisfy a simple condition

$$Q_0 \wedge Q_1 \wedge Q_2 \wedge Q_i = 0, \quad \forall i \in \{3, \dots, N + 1\}. \tag{7}$$

Finally, note that the non-singular solution forms a 2-dimensional distribution which can be parametrized e.g. as follows:

$$\dot{q} = G \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}, \tag{8}$$

where  $G = (g_{ij})$  is a control matrix. Thus if we consider the snake robot configuration space with coordinates (1) as manifold  $M$ , the solution above forms a set of vector fields  $\{g_1, g_2, g_3, g_4\}$ , such that  $g_3, g_4 \in \mathcal{Z}$ .

It is clear, that the space  $\text{span}\{g_1, g_2\}$  determines the set of accessible spatial velocity vectors and thus, taking into account the vector field flows  $\exp(tg_1)$ ,  $\exp(tg_2)$ , the possible trajectories of the snake robot. On the other hand, due to non-commutativity of  $\exp(tg_1)$ ,  $\exp(tg_2)$ , the robot can move even along the flow of the Lie bracket by means of the composition

$$\exp(-tg_2) \circ \exp(-tg_1) \circ \exp(tg_2) \circ \exp(tg_1).$$

Extending this idea, the space  $D_q$  of all movement directions at the point  $q$  is given by all possible Lie brackets of  $g_1(q)$  and  $g_2(q)$  and the resulting vector fields. From the geometric control theory point of view, it is quite necessary that the dimension of  $D_q$  is equal to the dimension of the tangent space  $T_qM, q \in M$ , which in our case is 6,7 and 8, respectively. Note that this is the condition on the model local controllability given by the Rashevsky–Chow Theorem.

## 6 Notes on the inverse kinematics

Note that the geometric meaning of the inner product of two conformal vectors  $U$  and  $V$  is the following.

Table 1: Geometric meaning of the inner product

$U \cdot V$	Plane	Sphere	Point
Plane	Angle between planes	Euclidean distance from center	Euclidean distance
Sphere	Euclidean distance from center	Distance measure	Distance measure
point	Euclidean distance	Distance measure	Euclidean distance

In our case, we have the set of point pairs  $\{P_0, \dots, P_N\}$  which determines the mechanism configuration uniquely. Using the notation of Section 4, the set of the admissible points  $\{Q_0, \dots, Q_{N+1}\}$  with respect the set of point pairs  $\{P_0, \dots, P_N\}$  has to satisfy the equations

$$Q_i \cdot Q_{i+1} = 2, \tag{9}$$

where  $i \in \{0, \dots, N\}$ . A point in the configuration space  $M$  is determined by the following direct computation:

$$\begin{aligned} x &= Q_0 \cdot e_1, \\ y &= Q_0 \cdot e_2, \\ \cos \theta &= (P_1 \wedge e_\infty) \cdot (e_0 \wedge e_1 \wedge e_\infty), \\ \cos \phi_i &= (P_i \wedge e_\infty) \cdot (P_{i+1} \wedge e_\infty). \end{aligned}$$

Various inverse problems can be solved by these equations. For instance assume the position of the first and the last link is fixed (as it does not affect the control process) and all possible resulting configurations are computed. We shall use the property of the scalar product of a point  $Q$  and a sphere  $S$  that the number  $\sqrt{2|Q \cdot S|}$  determines the Euclidean distance of  $Q$  and the point of tangency on  $S$  appropriate to the tangent containing  $Q$ . To avoid the singular initial positions we suggest to employ the assumption for small  $\epsilon \in \mathbb{R}$

$$2 - \epsilon \leq Q_i \cdot (Q_{i+2} - 2e_\infty) \leq 2 + \epsilon,$$

which reads that the angle  $\phi_i$  is in the vicinity of  $\frac{\pi}{2}$  or less restrictive assumption

$$\text{sgn}((P_i \wedge e_\infty) \cdot (P_{i+1} \wedge e_\infty)) = -\text{sgn}((P_i \wedge e_\infty) \cdot (P_{i+1} \wedge e_\infty))$$

which reads that the angle signum is changing in each joint.

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